First Order Ordinary Differential Equations

1. First Order Linear ODE: $y' + f(x)y = g(x)$

   Integrating Factor: $\mu(x) = e^{\int f(x)dx}$ with $C = 0\,,\ \mu(x)y'(x) + \mu(x)f(x)y(x) = \mu(x)g(x) \Longrightarrow$

   $\frac{d}{dx}[\mu(x)y(x)] = \mu(x)g(x) \Longrightarrow \mu(x)y(x) = \displaystyle\int \mu(x)g(x)\,dx + C \Longrightarrow y(x) = \frac{1}{\mu(x)}\int \mu(x)g(x)\,dx + \frac{C}{\mu(x)}$

   Or, integrating factor: $\mu(x) = e^{\int_{x_0}^{x} f(t)dt}$ and $y(x) = \frac{1}{\mu(x)}\displaystyle\int_{x_0}^{x} \mu(t)g(t)\,dt + \frac{y(x_0)}{\mu(x)}$

2. First Order Separable ODE: $\dfrac{dy}{dx} = \dfrac{g(x)}{h(y)}$

   Implicit Solution: $\displaystyle\int h(y)\,dy = \int g(x)\,dx \Longrightarrow H(y) = G(x) + C$ with $H' = h$ and $G' = g$

   Or, $\displaystyle\int_{y(x_0)}^{y} h(t)\,dt = \int_{x_0}^{x} g(t)\,dt$

3. Exact ODE: $M(x,\,y) + N(x,\,y)\dfrac{dy}{dx} = 0$ is called exact if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

   Implicit Solution: $F(x,\,y) = C$ where $\dfrac{\partial F}{\partial x} = M$ and $\dfrac{\partial F}{\partial y} = N$

   Start with $\frac{\partial F}{\partial x} = M$ or $\frac{\partial F}{\partial y} = N$ integrate with respect to x or y, respectively, then differentiate with respect to the other variable, and use the other equation to find the remaining function of y or x.

4. Homogeneous ODE: $\dfrac{dy}{dx} = f\left(\dfrac{y}{x}\right)$

   Let $v = \dfrac{y}{x}$. Then $y = x\,v$, $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$, and $v + x\dfrac{dv}{dx} = f(v)$

   Solve the separable ODE $\dfrac{dv}{dx} = \dfrac{f(v) - v}{x}$ for v, replace v with $\dfrac{y}{x}$, and solve for y.

Second Order Ordinary Differential Equations

Methods of Undetermined Coefficients (or Annihilator Method), Laplace Transform, and Series Solutions are not included.

5. Second Order Linear ODE with Constant Coefficients: $ay'' + by' + cy = 0$

   Characteristic Equation: $ar^2 + br + c = 0$ with solutions r_1 and r_2

$$y(x) = \begin{cases} c_1 e^{r_1 x} + c_2 e^{r_2 x}, & \text{if } r_1 \text{ and } r_2 \text{ are real-valued and unequal} \\ c_1 e^{r_1 x} + c_2\, x\, e^{r_1 x}, & \text{if } r_1 = r_2 \\ c_1 e^{\lambda x} \cos \mu x + c_2 e^{\lambda x} \sin \mu x, & \text{if } r_1,\, r_2 = \lambda \pm \mu i \end{cases}$$

   If $r_1,\, r_2 = \pm r$, then $y(x) = c_1 e^{-rx} + c_2 e^{rx}$ or $y(x) = c_1 \cosh rx + c_2 \sinh rx$ or

   $y(x) = c_1 \cosh r(x - x_0) + c_2 \sinh r(x - x_0)$ or

   $y(x) = c_1 \sinh r(x - x_0) + c_2 \sinh rx$ or $y(x) = c_1 \cosh r(x - x_0) + c_2 \cosh rx$

6. Second Order Linear Nonhomogeneous ODE: $y'' + p(x)y' + q(x)y = g(x)$

   General Solution: $y(x) = y_h(x) + y_p(x)$ where the homogeneous solution $y_h(x) = c_1 y_1(x) + c_2 y_2(x)$ is the general solution to the homogeneous equation $y'' + p(x)y' + q(x)y = 0$, while y_1 and y_2 are two linearly independent solutions of the same homogeneous equation, and the particular solution $y_p(x)$ is a solution to the nonhomogeneous equation $y'' + p(x)y' + q(x)y = g(x)$.

   Method of Variation of Parameters: $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$ where $u_1'(x) = \frac{-y_2(x)g(x)}{W(x)}$, $u_2'(x) = \frac{y_1(x)g(x)}{W(x)}$ and the Wronskian $W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x)$.

$$y_p(x) = y_1(x) \int \frac{-y_2(x)g(x)}{W(x)}\, dx + y_2(x) \int \frac{y_1(x)g(x)}{W(x)}\, dx \text{ or}$$

$$y_p(x) = y_1(x) \int_{x_0}^{x} \frac{-y_2(t)g(t)}{W(t)}\, dt + y_2(x) \int_{x_0}^{x} \frac{y_1(t)g(t)}{W(t)}\, dt$$

7. Cauchy-Euler Equation: $x^2 y'' + \alpha x y' + \beta y = 0$

   Indicial Equation: $p(p-1) + \alpha p + \beta = 0$ with solutions p_1 and p_2

$$y(x) = \begin{cases} c_1 |x|^{p_1} + c_2 |x|^{p_2}, & \text{if } p_1 \text{ and } p_2 \text{ are real-valued and unequal} \\ (c_1 + c_2 \ln |x|)|x|^{p_1}, & \text{if } p_1 = p_2 \\ |x|^{\lambda}[c_1 \cos(\mu \ln |x|) + c_2 \sin(\mu \ln |x|)], & \text{if } p_1,\, p_2 = \lambda \pm \mu i \end{cases}$$